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CONVECTIVE INSTABILITY OF AIR IN SNOW

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Convective instability of air in the pores between ice crystals in snow is considered. In the Boussinesq approximation, a system of equations that describes the origin of thermal convection within the snow thickness is derived. It is shown that for snow, as for a liquid, there is a criterial number analogous to the Rayleigh number that determines the origin of air instability in snow. The contribution of natural convection of air to the heat- and mass-transfer processes in snow is estimated and possible reasons for the considerable scatter of reported experimental data on the thermal conductivity and diffusivity of snow are discussed.

1. Thermal Conductivity and Diffusivity of Snow. We consider a snow with macroscopically stationary contained air, which fills the pores between the ice crystals composing the snow skeleton. In the presence of a temperature gradient, the steam migration in snow from more heated to less heated places occurs. The steam transfer proceeds via molecular diffusion, and the macroscopic gradient of steam concentration depends solely on the temperature gradient. Indeed, the duration of steam-concentration equalization in the pores between the ice crystals is $\tau = \delta^2/D_n$, where δ is the typical size of the pores and D_n is the diffusivity of water molecules in air, which for $\delta = 0.1$ cm and $D_n = 0.2$ cm^2/sec gives about 0.05 sec. Thus, the steam in snow pores under actual conditions is in a near-saturated state, and, hence, its concentration is a function of only one parameter, the temperature.

Based on the Clausius–Clapeyron equation, the approximate dependence of the saturating steam density ρ_n on the temperature θ (in Celsius degrees) is given by the formula

$$\rho_n \approx \rho_n^*\Big(1 - L\theta/(R_n T_0^2)\Big), \tag{1.1}$$

where ρ_n^* is the saturating steam density at 0°C, $T_0 = 273$ K, R_n is the gas constant of the steam, and L is the specific heat of steam sublimation.

The heat flux through unit area is $j \sim n^{2/3}\delta^2$, where n is the number of pores in a unit volume related to the snow porosity f by the formula $n = f/n^3$. Hence, we have $j \sim f^{2/3}$. With allowance for this dependence, the heat flux through unit area can be written as $\boldsymbol{q} = -(\lambda_s\nabla\theta + f^{2/3}LD_n\nabla\rho_n) = -\lambda^*\nabla T$. Here λ_s is the thermal conductivity of snow, f is the snow porosity, ∇ is the gradient operator, and λ^* is the effective thermal conductivity of snow:

$$\lambda^* = \lambda_s\Big(1 + L\rho_n^* f^{2/3} D_n/(R_n T_0^2 \lambda_s)\Big). \tag{1.2}$$

For $f = 1$, formula (1.2) coincides with the analogous formula obtained by Sulakvelidze [1].

By writing the equations of thermal conduction and steam diffusion in snow with allowance for relation (1.1), it can be easily shown that the effective diffusivity D^* of steam in snow coincides with its thermal diffusivity $\chi^* = \lambda^*/(\rho c)_s$, where $(\rho c)_s$ is the heat capacity of snow per unit volume. From here, it follows that the field of concentration of steam molecules in snow is determined by the temperature field, and the process of steam diffusion in snow is identical to heat transfer.

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It should be noted that, by inserting typical values of the parameters into the right side of equality (1.2), one can easily find that the second term in parentheses equals only several hundredth parts of unity. From here, it follows that, in the absence of convection, the contribution of the condensation processes to thermal conduction and diffusion in snow is insignificant.

The calculated diffusivities D^* for different types of snow amount to $4 \cdot 10^{-3}$ cm^2/sec [2]. Nevertheless, numerous experimental data on D^* obtained both in laboratory and field conditions are within the range from 0.13 to 1.1 cm^2/sec and in most cases exceed the diffusivity of steam molecules in air [3].

Such abnormally high values of thermal conductivity and diffusivity of steam in snow can be explained by the fact that the convection in air contained in snow was involved in these experiments.

2. Equations of Thermal Convection in Snow. Snow is a porous medium, which consists of a rigid skeleton formed by randomly located ice crystals and pores filled with air, which can freely flow through them both inward and outward. In the presence of a temperature gradient, a convective flow of the air filling the pores may arise in snow. In the filtration theory, the macroscopic filtration velocity $\boldsymbol{u}$ is used. It is related to the mean velocity of air particles in the porous medium $\boldsymbol{V}$ by the relation $\boldsymbol{u} = f\boldsymbol{V}$.

The macroscopic flow of air in snow is described by a system of fluid-dynamic equations that include the continuity equation, the equation of motion, and the heat-transfer equation.

By analogy with the equations of motion for damp soil with account of friction between the solid and liquid phases [4], the equation of motion of air in snow can be written as

$$f\rho \frac{d\boldsymbol{V}}{dt} = -f\nabla p + f\rho \boldsymbol{g} + \boldsymbol{F},$$

where ρ is the air density, $\boldsymbol{g}$ is the free-fall acceleration, and $\boldsymbol{F}$ is the force exerted on unit mass of air by the porous medium. According to the Darcy law, this force normalized to unit volume of snow equals $f\mu\boldsymbol{V}/\sigma$, where μ is the air viscosity and σ is the snow permeability.

Thus, the equation of motion acquires the form

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\nabla)\boldsymbol{V} = -\frac{1}{\rho}\nabla p + \boldsymbol{g} - \frac{\nu}{\sigma}\boldsymbol{V}, \tag{2.1}$$

where ν is the kinematic viscosity of air.

To properly write the equation of heat inflow, we note that the condensation and evaporation processes proceeding with participation of the steam contained in snow and accompanying the flow of air through the snow skeleton are a distinctive feature of snow compared to water-saturated sand soil. These processes must be taken into account in formulating the heat-balance equation.

Next, we assume that the air that flows through the snow all the time remains saturated. Then, the total heat flux through unit area $\boldsymbol{j}_T$ caused by the conductive heat transfer through the snow skeleton and the heat transfer through the pores filled with the steam–air mixture can be represented in the form $\boldsymbol{j}_T = -\lambda^*\nabla T + f\rho c_p\theta\boldsymbol{V} + Lf\rho_n\boldsymbol{V}$, where c_p is the heat capacity of air at constant pressure.

In this case, the heat-transfer equation acquires the form

$$\frac{\partial\theta}{\partial t} + M\boldsymbol{V}\nabla\theta = \chi\Delta\theta, \tag{2.2}$$

where Δ is a Laplacian and $M = \dfrac{f\rho c_p}{\rho_s c_s}\Big(1 + \dfrac{L^2\rho_n^*}{R_n T_0^2 \rho c_p}\Big)$ is a coefficient.

Finally, the continuity equation is

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{V}) = 0. \tag{2.3}$$

System (2.1)–(2.3) must be supplemented by the equation of state of the medium $\rho = \rho(T, p)$.

We consider small deviations from the equilibrium state that are described by the temperature, pressure, and density perturbations (θ', p', and ρ', respectively) and by the velocity $\boldsymbol{V}$. Then the linear system of equations for thermal convection in snow in the Boussinesq approximation [5] is

$$\frac{\partial\boldsymbol{V}}{\partial t} = -\frac{1}{\rho_0}\nabla p' - \frac{\nu}{\sigma}\boldsymbol{V} + \alpha g\theta'\boldsymbol{e}_z, \quad \frac{\partial\theta'}{\partial t} + M(\boldsymbol{V}\nabla\langle\theta\rangle) = \chi^*\Delta\theta', \quad \operatorname{div}\boldsymbol{V} = 0, \tag{2.4}$$

where $\langle\theta\rangle$ is given by the equation $\Delta\theta = 0$, $\boldsymbol{e}_z$ is the unit vector directed vertically upward, and α is the coefficient of thermal expansion of air.

In what follows, we restrict ourselves to the case where the snow is situated between two horizontal planes $z = 0$ and $z = H$ that have temperatures θ_0 and θ_1, respectively, and $\theta_0 > \theta_1$. In this case, the stationary temperature field is described by the formula $\langle\theta\rangle(z) = \theta_0 - z(\theta_0 - \theta_1)/H$.

With H, σ/ν, and $\chi^*/(HM)$ chosen as the unit length, time, and velocity, respectively, and pressure and temperature perturbations measured in units of $\rho_0\nu\chi^*/\sigma$ and $\theta_0 - \theta_1$, the dimensionless form of system (2.4) is

$$\frac{\partial \boldsymbol{V}}{\partial t} = -M\nabla p + \mathrm{R}^*\theta \boldsymbol{e}_z - \boldsymbol{V}; \tag{2.5}$$

$$\mathrm{Pr}^* \frac{\partial \theta}{\partial t} = \boldsymbol{V}\boldsymbol{e}_z + \Delta\theta, \qquad \operatorname{div} \boldsymbol{V} = 0 \tag{2.6}$$

(the primes are omitted). Here, $\mathrm{R}^* = \alpha g\sigma HM(\theta_0 - \theta_1)/(\nu\chi^*)$ is a dimensionless parameter, which is an analog of the Rayleigh number [6], and $\mathrm{Pr}^* = (\nu/\chi^*)(H^2/\sigma)$ is an analog of the Prandtl number.

Thus, the natural convection in the snow bed is characterized by two dimensionless parameters, R^* and Pr^*.

Following the traditional procedure of [5], we eliminate the pressure p and the horizontal velocities V_x and V_y from Eqs. (2.5) and (2.6). To this end, we apply the operator rot (rot) to Eq. (2.5) and project the resulting equation onto the z axis. As a result, we obtain

$$\frac{\partial}{\partial t}\Delta V_z = \mathrm{R}^*\Delta_1\theta + \Delta V_z, \qquad \mathrm{Pr}^* \frac{\partial \theta}{\partial t} = \Delta\theta + V_z, \tag{2.7}$$

where $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is a planar Laplacian.

Equations (2.7) allow partial solutions that describe the so-called normal disturbances exponential in time and periodical in the plane (x, y):

$$V_z(x, y, z) = V_0(z)\exp[\lambda t + i(k_1 x + k_2 y)], \quad \theta(x, y, z) = \theta_0(z)\exp[\lambda t + i(k_1 x + k_2 y)]. \tag{2.8}$$

Substituting (2.8) into (2.7), we obtain the following amplitude equations:

$$\lambda(V_0'' - k^2 V_0) = V_0'' - k^2 V_0 + k^2\mathrm{R}^*\theta_0, \quad \lambda\theta\mathrm{Pr}^* = \theta_0'' - k^2\theta_0 + V_0. \tag{2.9}$$

Here $k^2 = k_1^2 + k_2^2$ and the primes denote differentiation with respect to z.

For equations (2.9), boundary conditions should be set, which reflect particular physical conditions. Below, some simplest types of such conditions are considered.

(1) At both surfaces of the snow bed, fixed temperatures are set; hence, the temperature disturbances at them vanish: $\theta(z)\big|_{z=0.1} = 0$; moreover the upper and lower boundaries are impermeable: $V_z(z)\big|_{z=0.1} = 0$. Solutions of the types $\theta_0(z) = A_n \sin(\pi n z)$ and $V_0(z) = B_n \sin(\pi n z)$ $(n = 1, 2, \ldots)$ satisfy these conditions.

Substituting the above solutions into (2.9), we obtain a system of equations in A_n and B_n. The consistency condition for this system reduces to the equality to zero of its determinant, which yields $\lambda^2 + B\lambda - C = 0$, where $B = \pi^2 n^2 + k^2$ and $C = \pi^2 n^2 - k^2\mathrm{R}^* M/(\pi^2 n^2 + k^2)$. In this case, the condition of growth of the disturbance amplitude $\operatorname{Re}\lambda > 0$ reduces to the inequality $C > 0$; this means that instability arises when the condition $\mathrm{R}^{**} = M\mathrm{R}^* > (\pi^2 n^2 + k^2)^2/k^2$ is fulfilled, i.e., the stability thresholds depend on n and k.

For a given n, those disturbances are most unstable for which the function $\mathrm{R}^*(k, n)$ has a minimum, $\partial\mathrm{R}^*/\partial k = 0$, which gives $k_m = \pi n$. To this value of the wavenumber, $\mathrm{R}^*(n) = 4\pi^2 n^2$ corresponds. With the growth of R^{**}, the disturbances with $n = 1$ and wavenumber $k = \pi$ become most unstable, and the threshold number R_m^{**} at which instability emerges is $4\pi^2$.

Thus, the condition for the appearance of convection in a snow bed of thickness H at a given temperature difference $\Delta\theta = \theta_0 - \theta_1$ reduces to the inequality

$$\alpha MHg\sigma\Delta\theta/(\chi^*\nu) > 4\pi^2. \tag{2.10}$$

This result corresponds to the case where the snow surface is covered by a thin film impermeable to air. Under natural conditions, this situation is widely met. For example, after a relatively warm sunny day on the onset of the night chill, snow crystals often freeze together, forming a thin transparent ice crust that arrests the damp air inside the snow bed. With further cooling, the temperature gradient in the snow increases, thus promoting the onset of air convection in it.

Condition (2.10) can be represented in the following more convenient form:

$$\Delta\theta/H > 4\pi^2\chi^*\nu/(\alpha M H^2 g\sigma). \tag{2.11}$$

If the temperature gradient in the snow bed satisfies condition (2.11), air convection originates in it.

(2) We assume that the snow bed is permeable to air, i.e., the air freely escapes from the snow bed, rising upward. In this case, the horizontal velocities V_x and V_y may be thought of as vanishing at the upper surface of the snow bed; hence, by virtue of the relation $k_1V_x + k_2V_y = (1/i)(dV_z/dz)$, which follows from the continuity equation, we have $V_z'(1) = 0$. In addition, $V_z(0) = 0$. We also assume that $\theta(0) = 0$ and $\theta'(1) = 0$. It is apparent that solutions of the type

$$\theta(z) = A_n \sin\frac{\pi(2n+1)z}{2}, \qquad V_z(z) = B_n \sin\frac{\pi(2n+1)z}{2} \qquad (n = 0,\ 1,\ 2, \ldots)$$

satisfy the conditions.

The above results remain valid on substituting $(2n+1)/2$ for n. In this case, $\mathrm{R}_m^{**} = 4\pi^2((2n+1)/2)^2$.

The disturbances with $n = 0$ and transverse wavenumbers $(\pi/2, 0)$ and $(0, \pi/2)$ are the most unstable. The critical Rayleigh number is $\mathrm{R}_k^{**} = \pi$. Instead of condition (2.10), we obtain

$$\alpha M H g\sigma\Delta\theta/(\chi^*\nu) > \pi^2. \tag{2.12}$$

From here, it follows that, for the open surface of the porous medium, the critical temperature difference for the onset of convection is four times as low as that for the medium covered by an impermeable film, the critical wavelength being twice as high.

(3) The boundary conditions $V_z(0) = 0$, $V_z'(1) = 0$, $\theta(0) = 0$, and $\theta(1) = 0$ are of particular interest. In this case, the maximum velocity is attained when the particles leave the snow bed. Here, no simple solutions exist analogous to those described above. The approximate solution to the problem can be obtained numerically using the Bubnov–Galerkin method.

Since for $\lambda = 0$ the curve $\mathrm{R}^*(n, k)$ is a neutral curve, which separates the regions of unstable and stable disturbances, assuming $\lambda = 0$ in amplitude equations (2.9), we obtain the following boundary-value problem for neutral disturbances:

$$(V'' - k^2V) + k^2\mathrm{R}^*\theta = 0; \tag{2.13}$$

$$(\theta'' - k^2\theta) + V = 0; \tag{2.14}$$

$$V_z(0) = V_z'(1) = 0; \tag{2.15}$$

$$\theta(0) = \theta(1) = 0. \tag{2.16}$$

The critical Rayleigh numbers R_k^* are the eigenvalues of problem (2.13)–(2.16) and the amplitudes of critical disturbances are its eigenfunctions.

For the velocity $V_z(z)$ we adopt the simplest approximation $V_z = \sin(\pi nz/2)$. In this case, conditions (2.15) are automatically fulfilled. The solution of Eq. (2.14) that satisfies boundary conditions (2.16) is

$$\theta(z) = \frac{1}{k^2 + (\pi/2)^2}\Big(\sin\frac{\pi z}{2} - \frac{\sinh(kz)}{\sinh k}\Big). \tag{2.17}$$

Inserting (2.17) into (2.13), multiplying both parts of the resulting equation by $V(z)$ and integrating it between the limits from 0 to 1, we obtain

$$\mathrm{R}^{**} = \frac{(k^2 + \pi^2/4)^2}{k^2(1 - 2k\coth k/(k^2 + \pi^2/4))}. \tag{2.18}$$

As calculations show, the minimum critical Rayleigh numbers for the fundamental disturbance ($n = 1$) is $R_m^{**} = 28.2$ and it is attained for the wavenumber $k_m = 2.4$.

Thus, in all the above cases of possible convection modes in snow, the critical Rayleigh number R_k^* lies between π^2 and 4π. These values coincide with the critical Rayleigh numbers for sand saturated with water, which were obtained under the assumption that the heat capacity per unit volume of the liquid $(\rho c)_*$ equals the heat capacity per unit volume of the porous medium ρc [5]. For snow, the values of R_k must be multiplied by $M \sim 10^3$.

For the snow density $\rho = 0.3$ g/cm^3, we obtain the parameter $M = 1.3 \cdot 10^{-3}$ and the coefficient $\chi^* = 2.7 \cdot 10^{-3}$ cm^2/sec. In this case, condition (2.12) acquires the form $\Delta\theta > 0.66/(kH)$.

Unfortunately, we have no experimental data on the snow permeability σ for different types of snow. However, on the basis of some model concepts about a porous medium, Shaidegger proposed the theoretical formula $\sigma = f\delta^2/32$ [7]. There is also the Kozeny semi-empirical formula that has the form $\sigma = f^3\delta^2/(150(1-f)^2)$ in our notation. For $f > 0.6$, both formulas give practically identical results. Using them, for the snow depth $H = 40$ cm, we obtain $\Delta\theta > 16$°C and $\Delta\theta > 4$°C for $\delta = 0.2$ cm and $\delta = 0.4$ cm, respectively.

From these estimates, it follows that air convection in snow (in the absence of wind) is possible only for a large thickness and low density of the snow bed, e.g., when night chill sets in at dusk on a relatively warm day. However, as follows from the theory of propagation of heat waves in snow, daily temperature variations can reach only a relatively small depth in the snow (about 10–15 cm). Therefore, the effect of wind ignored, the origination of air convection in a heavy snow bed seems to be highly improbable.

To estimate the order of magnitude of the velocity $V_z(z)$, we represent $\Delta\theta$ as the sum of two components, $\Delta\theta = \Delta\theta_k + \Delta\theta_1$, where $\Delta\theta_k$ is the critical temperature difference for the onset of air instability given by condition (2.11) or (2.12), and $\Delta\theta_1$ is an additional (supercritical) temperature difference responsible for sustaining a steady convective flow of air in snow. The steady air velocity in the snow bed V is of order $(\beta g\sigma/\nu)\Delta\theta_1$, which, for $\sigma \approx 10^{-3}$ cm^2 and $\Delta\theta_1 = 1$°C, roughly amounts to $4 \cdot 10^{-2}$ cm/sec.

The convective air flow affects the heat- and mass-transfer processes in snow differently. The effective thermal conductivity of snow due to the air flow is $\langle\lambda\rangle^* \sim \rho c_p VH$, which for the above parameters gives about $4 \cdot 10^{-4}$ cal/(cm $\cdot$ sec $\cdot$ deg), i.e., it is of the same order as λ^*. The effective diffusivity is $\langle D\rangle^* \sim VH = 1.6$ cm^2/sec, which is significantly greater than that of steam in snow without convection.

These estimates show that the influence of air convection in snow is greater. This conclusion also originates from the following reasoning. The dimensionless Peclet number that characterizes the ratio of the convective term in the diffusion equation to the inductive term is $\mathrm{Pe} = \langle V\rangle H/\langle D\rangle^* = \langle V\rangle H/\chi^*$, where $\langle V\rangle$ is the mean flow velocity. The analogous Peclet number for heat-transfer processes is $\mathrm{Pe}^* = f\rho c_p\langle V\rangle H/(\rho_s c_s \chi^*) = f(\rho c_p/(\rho_s c_s))\mathrm{Pe}$, i.e., $\mathrm{Pe}^* \approx 10^{-3} \cdot \mathrm{Pe}$.

Thus, the relative role of air convection in the mass-transfer processes proceeding in snow is more important than that in thermal conduction, which qualitatively agrees with numerous experimental data. The fact that these experimental data are rather contradictory and demonstrate a considerable scatter can be explained by the probable occurrence of convective air flows in some experiments both with snow samples and with snow beds.

From the above, it is apparent that, in conducting calculations on water balance and evaluating soil freezing depths, one must take into account the possibility of air convection in snow beds and its contributions to heat- and mass-transfer processes.

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